Homework 9, due 11/28

1. Consider the cover $\mathfrak{U} = \{U_1, U_2\}$ of $\mathbf{P}^1$ given by $U_1 = \mathbf{P}^1 \setminus \{\infty\}$ and $U_2 = \mathbf{P}^1 \setminus \{0\}$. Show that $H^1(\mathfrak{U}, \mathcal{O}) = 0$.

2. (a) Show that dz defines a meromorphic one-form on $\mathbf{P}^1$, with no zeros, and a double pole at ∞.

   (b) Let α be any meromorphic one-form on $\mathbf{P}^1$. Show that

   $$\sum_{p \in \mathbf{P}^1} \mathrm{ord}_p \alpha = -2.$$

   Hint: show that $\alpha = f dz$ for a meromorphic function f.

   (c) Let $p_1, \ldots, p_k \in \mathbf{P}^1$, and $a_1, \ldots, a_k \in \mathbf{Z}$, satisfying $\sum_i a_i = -2$. Can you find a meromorphic one-form α on $\mathbf{P}^1$ such that $\mathrm{ord}_{p_i} \alpha = a_i$ for each i, and $\mathrm{ord}_p \alpha = 0$ for all other p?

3. Consider the one-form $\alpha = \bar{z} dz$ on $\mathbf{C}$.

   (a) Does there exist a function $f : \mathbf{C} \to \mathbf{C}$ such that $\alpha = df$?

   (b) Does there exist $f : \mathbf{C} \to \mathbf{C}$ such that $\alpha = \partial f$?

4. Let $X = \mathbf{C}/\Lambda$ be a complex torus, where $\Lambda = \{m_1 w_1 + m_2 w_2 : m_1, m_2 \in \mathbf{Z}\}$ for $w_1, w_2 \in \mathbf{C}$, and $\mathrm{Im}\,(w_1/w_2) > 0$.

   (a) Recall that $dz, d\bar{z}$ define one-forms on X. Compute

   $$\int_X dz \wedge d\bar{z}.$$

   (b) Suppose that α is a meromorphic one-form on X. Show that

   $$\sum_{p \in X} \mathrm{ord}_p \alpha = 0.$$

   (c) Just as in question 2(c), suppose that $p_1, \ldots, p_k \in X$, and $a_1, \ldots, a_k \in \mathbf{Z}$ satisfy $\sum_i a_i = 0$. Is there a meromorphic one-form α on X such that $\mathrm{ord}_{p_i} \alpha = a_i$ for each i, and $\mathrm{ord}_p \alpha = 0$ for all other p?

5. Suppose that α is a (1,0)-form on a compact Riemann surface X.

   (a) If in a local holomorphic chart $\alpha = \alpha_z dz$, define $\overline{\alpha} = \overline{\alpha_z} d\bar{z}$. Show that $\overline{\alpha}$ defines a (0,1)-form on X, i.e. check that the coordinate representations of $\overline{\alpha}$ satisfy the right compatibility condition.

   (b) Show that

   $$\int_X \frac{i}{2} \alpha \wedge \overline{\alpha} \geq 0,$$

   with equality only if $\alpha = 0$.